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Translated by C.A.M.

PMM U.S.S.R.,Vol.51,No.5,pp.626-634,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00
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NON-LOCAL NON-LINEAR EQUATIONS OF WIND WAVES OVER AN UNEVEN BOTTOM*

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The evolution of a two-layer water-air medium under the action of a wind is treated in the weak non-linearity approximation. Here, together with the effects studied in /1-3/, we present, using an operator method /4, 5/, analogies of the Boussinesq equations without any assumption regarding the shallowness of the water reservoir and also taking account of the action of a wind but under the assumption that the amplitudes of the corresponding wave processes are small and the average velocity of the wind and the bottom of the reservoir are specified functions which vary "slowly" with the horizontal coordinates and time. Non-local (pseudo-differential) equations are obtained which describe the behaviour of the medium being studied taking account of the quadratic and cubic non-linear terms. Asymptotic solutions of these equations which take account of weak resonance interactions are constructed using the methods in /6, 7/. Algorithms are given for deriving the analogous equations and the construction of their asymptotic solutions when account is taken of an arbitrary degree of non-linearity.

**Prikl.Matem.Mekhan.*,51,5,798-806,1987

1. Reduction of the initial problem to a problem on the free boundary. The evolution of a two-layer water-air medium under the action of a wind in a reservoir with an uneven bottom is considered under the following assumptions: the water and air are incompressible, non-viscous, the velocity field in the air consists of a specified average velocity and perturbations which are small compared with this velocity, $V = (U_1 + u_1,\ U_2 + u_2,\ w_2)$, and the motion of the water is described by a velocity potential $\Phi(x, z, t)$. Here t is the time, $x = (x_1, x_2)$ are the horizontal coordinates, z is the vertical coordinate and $z = 0$ corresponds to the unperturbed boundary of separation between the two media. The bottom of the water reservoir is described by the equation $z = -D(x)$ while the free boundary is described by the equation $z = \eta(x, t)$. The function $D(x) > 0$ is specified.

The initial conditions consist of the equations for the conservation of mass and momentum and the following boundary conditions: on the bottom of the water reservoir (the solid wall), on the boundary of separation (dynamic and kinematic) and on the upper boundary of the driving layer (the perturbations are negligibly small as $z \to \infty$). The evolution of the medium being studied is considered at times $\sim L/\sqrt{g\lambda}$, where $\lambda \sim U_\infty^2/g$, $L\,(h = \lambda/L \ll 1)$ are the characteristic wavelength and acceleration length for the wave processes being studied, U_∞ is the average velocity of the wind on the upper boundary of the driving layer and g is the acceleration due to gravity. The dimensions of the water reservoir are assumed to be much greater than L which excludes the influence of bank effects from the treatment.

Let us introduce the dimensionless parameters $\delta = \rho_2/\rho_1 \ll 1$ and $\varepsilon = \eta_0/\lambda$ which describe the ratio of the densities of air and water and the characteristic inclination of the waves. Here η_0 is the characteristic amplitude of the swell studied and characterizes the nonlinearity, which is assumed to be small and associated with δ by means of the relationship $\delta = \varepsilon^2$. We assume that the relief of the bottom of the water reservoir and the mean velocity of the wind are slowly varying functions of x and t:

$$|\nabla D| \sim \frac{\lambda}{L},\ |\nabla U_i| \sim \frac{\sqrt{g\lambda}}{L},\ \frac{\partial U_i}{\partial z} \sim \sqrt{\frac{g}{\lambda}}$$
$$\frac{\partial U_i}{\partial t} \sim \frac{g\lambda}{L},\ \ i = 1, 2,\ \ \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$$

In the dimensionless variables $x' = x/L,\ z' = z/\lambda,\ t' = t\sqrt{g\lambda}/L$, the system of equations and boundary conditions describing the evolution of the medium being studied take the form, after the pressure has been eliminated,

$$\begin{aligned}
&z = \varepsilon\eta(x, t) && (1.1)\\
&h\nabla(\eta + h\Phi_t + {}^1/_2\varepsilon\,|h\nabla\Phi|^2 + {}^1/_2\varepsilon\Phi_z^2) + \\
&\quad \varepsilon h\nabla\eta\,(\partial/\partial z)\,(h\Phi_t + {}^1/_2\varepsilon\,|h\nabla\Phi|^2 + {}^1/_2\varepsilon\Phi_z^2) = \\
&\quad \varepsilon^2\{(1 + \varepsilon h w_t + \varepsilon^2 h\nabla w\cdot u + \varepsilon^2 w_z w)\,h\nabla\eta + hu_t + \\
&\quad \varepsilon\,(u\cdot h\nabla)\,u + U_z w + \varepsilon u_z w\}\\
&h\eta_t + \varepsilon h\nabla\eta\cdot u = w\\
&h\eta_t + \varepsilon h\nabla\eta\cdot h\nabla\Phi = \Phi_z\\
&z = -D(x),\ \Phi_z + h\nabla D\cdot h\nabla\Phi = \Phi_z && (1.2)\\
&\varepsilon\eta(x, t) \leqslant z < \infty && (1.3)\\
&hu_{tz} + (\partial/\partial z)\,\{(U\cdot h\nabla)\,u + \varepsilon\,(u\cdot h\nabla)\,u + (u\cdot h\nabla)\,U + \\
&\quad U_z w + \varepsilon w u_z\} = h^2\nabla w_z + h\nabla\,(U\cdot h\nabla w + \varepsilon u\cdot h\nabla w + \varepsilon w_z w)\\
&\varepsilon\eta(x, t) \leqslant z < \infty,\ h\nabla\cdot u + w_z = 0 && (1.4)\\
&-D(x) \leqslant z \leqslant \varepsilon\eta(x, t),\ h^2\Delta\Phi + \Phi_{zz} = 0 && (1.5)\\
&z \to \infty,\ u_i,\ w \to 0,\ i = 1, 2 && (1.6)\\
&\eta'(x', t') = \eta(x, t)/\eta_0,\ \Phi'(x', z', t') = \Phi(x, z, t)/(\eta_0\sqrt{g\lambda}),\\
&U'(x', t', z') = U(x, t, z)/\sqrt{g\lambda},\ (u', w')(x', t', z') = \\
&\quad (u, w)(x, t, z)/(\eta_0\sqrt{g/\lambda}),\ D'(x') = D(x)/\lambda
\end{aligned}$$

Here, $u = (u_1, u_2)$, $U = (U_1, U_2)$, $\Delta = \nabla^2$, $a_1\cdot a_2$ is a scalar product and the primes are omitted. The condition of "slowness" in the variation of the relief of the bottom and the average wind velocity in dimensionless variables means that the functions $D(x)$ and $U(x, t, z)$ are smooth and independent of h.

Let us consider the Cauchy-Poisson problem for system (1.1)-(1.6), that is, it is assumed that Φ, Φ_t, u and w on the free boundary are specified when $t = 0$ and we shall study solutions which are bounded in $R_x^2 \times R_t^+$ and smoothly dependent on ε and irregularly on h so that

$$\frac{\partial^{|\alpha|+\beta+\gamma}}{\partial x^\alpha \partial z^\beta \partial t^\gamma} \sim h^{-|\alpha|-\gamma},\quad |\alpha|, \beta, \gamma = 0, 1, \ldots$$

on the functions Φ, u, w and η.

In the investigation of wave motions of the surface of a liquid, the natural procedure involves the elimination of the variable z and the reduction of the problem to an investigation of the equatins for the boundary of separation $\eta\,(x, t)$ and the functions characterizing the potential Φ and the velocity in the air layer u, w in the variables $x = (x_1, x_2)$, t. These functions can be introduced unambiguously. For example, Φ, u and w can be specified when $z = 0$ or the mean values of Φ, u and w in the corresponding layers, etc.

In the case of the scheme being described here it is found to be most convenient to introduce the functions on the free boundary:

$$\varphi = \Phi\,|_{z=\varepsilon\eta}, \quad v = w\,|_{z=\varepsilon\eta}, \quad \psi = u\,|_{z=\varepsilon\eta} \tag{1.7}$$

When there is no upper layer the functions φ and η for the corresponding system constitute the so-called canonical variables /8/.

The elimination procedure for long wavelengths with weak dispersion without taking account of the wind is well-known and leads, under the assumptions that $\varepsilon = \sqrt{h}$, $D = \sqrt{h}D_1$ in the zeroth approximation with respect to h, to the wave equation in the next approximation to the Boussinesq equation and so on /9/. In the general case the execution of such a procedure without additional assumptions is, apparently, impossible. The assumption regarding the "slowness" of the variation in the relief of the bottom and the mean wind velocity makes it possible to carry out such a procedure and to reduce the problem to an investigation of a simpler system of equations for φ, η, ψ and v and then to a scalar and even a pseudodifferential equation for φ which leads to a substantial reduction in the volume of calculations compared with the initial problem (at least, in the asymptotic analysis). Such a procedure has been carried out in /10/*.(*Without taking account of the wind, in the paper: Dobrokhotov S.Yu., Multiphase asymptotics and Maslov's theory in the linear and non-linear equations of water waves, Voronezh, 1984, Deposited in the All-Union Institute for Scientific and Technical Information (VINITI), 3.07.84; No.4585-84). A similar approach was employed in /11, 12/ in the proof of the existence and uniqueness of the solutions which describe surface waves without allowing for the wind.

Since it is impossible to obtain exact formulae in the case of variable coefficients, we are only concerned here with expansions with respect to the parameter ε with an accuracy which has been specified beforehand. In order to take account of the quadratic and cubic terms, it suffices to carry out the procedure for eliminating z up to terms of the order of ε^2, and all subsequent formulae are therefore presented with an accuracy $O(\varepsilon^3)$. The final equations are only considered with this degree of accuracy for the functions φ and η and have the form (a scheme for their derivation is presented in Sect.3)

$$\begin{aligned} &h^2\Delta\,\{\eta + h\varphi_t + {}^1/_2\varepsilon\,(\,|\,h\nabla\varphi\,|^2 - (B_0\varphi)^2) - \\ &\quad \varepsilon^2\,(B_0\varphi\,(\eta\,(-h^2\Delta\varphi) - B_0\eta B_0\varphi))\} = \varepsilon^2\,\{h^2\Delta\eta - \\ &\quad h\,(\partial/\partial t)\,(B_1 h\eta_t) + (h\nabla\cdot U_z^\circ)\,h\eta_t + U_z^\circ\cdot h^2\nabla\eta_t\} + O\,(\varepsilon^3) \end{aligned} \tag{1.8}$$

$$\begin{aligned} &h\eta_t - B_0\varphi + \varepsilon\,\{h\nabla\cdot\eta h\nabla\varphi + B_0\eta B_0\varphi\} - \varepsilon^2\,\{B_0\varphi\,|\,h\nabla\eta\,|^2 + \\ &\quad B_0\,(\eta B_0\eta B_0\varphi + {}^1/_2\eta^2 h^2\Delta\varphi) + \eta h^2\Delta\,(\eta B_0\varphi) - {}^1/_2\eta^2 \times \\ &\quad h^2\Delta B_0\varphi\} = O\,(\varepsilon^3) \end{aligned} \tag{1.9}$$

Here B_0 and B_1 are pseudodifferential operators and their symbols have the form

$$B_0 = \frac{\partial R}{\partial z}(x, p, 0, h), \quad B_1 = \frac{\partial R_w}{\partial z}(x, t, p, \omega, 0, h)$$
$$R_w = R_{w_0} - ihR_{w_1} + \ldots$$

The function R_{w_0} is determined from the Rayleigh problem (3.3), $U_z^\circ = (\partial U/\partial z)\,|_{z=0}$ and the notation $B_0\eta B_0\varphi$ denotes $B_0\,[\eta B_0\varphi]$.

On the left-hand side of Eq.(1.8) which is predominant with respect to the parameter ε, the occurrence of the operator $h^2\Delta$ may, generally speaking, lead to the appearance of additional solutions compared with the initial problem or to the disappearance of some of the solutions. This, however, does not occur if only bounded φ and η are considered.

Let us now pass from system (1.8), (1.9) to a consideration of a single equation in the function φ. After applying the operator $h^2\Delta$ to (1.9) and the expressions $h^2\Delta\eta$ from (1.8) with an accuracy $O(\varepsilon^3)$ and η from (1.8) with an accuracy $O(\varepsilon^2)$ Eq.(1.9) is transformed into an equation for φ:

$$\begin{aligned} &h^2\Delta\,\{[h^2\varphi_{tt} + B_0\varphi] + \varepsilon\,[{}^1/_2\,(h\partial/\partial t)\,(\,|\,h\nabla\varphi\,|^2 - (B_0\varphi)^2) + \\ &\quad h\nabla\cdot\,(h\varphi_t h\nabla\varphi) + B_0 h\varphi_t B_0\varphi] + \varepsilon^2\,[(h\partial/\partial t)\,(B_0\varphi\,(h\varphi_t h^2\Delta\varphi + \\ &\quad B_0 h\varphi_t B_0\varphi)) + {}^1/_2 h\nabla\cdot(\,|\,h\nabla\varphi\,|^2 - (B_0\varphi)^2)\,h\nabla\varphi + \\ &\quad B_0\,(\,|\,h\nabla\varphi\,|^2 - (B_0\varphi)^2)B_0\varphi + (B_0\varphi\,|\,h^2\nabla\varphi_t\,|^2 - \end{aligned} \tag{1.10}$$

$$B_0 (h\varphi_t B_0 h\varphi_t B_0 \varphi - {}^1/_2 (h\varphi_t)^2 (-h^2\Delta\varphi)) +$$
$$h\varphi_t (-h^2\Delta (h\varphi_t B_0\varphi)) + {}^1/_2 (h\varphi_t)^2 h^2\Delta B_0\varphi]\} +$$
$$\varepsilon^2 \{h^2\Delta h\varphi_t - (h\partial/\partial t) B_1 h^2\varphi_{tt} + (h\nabla\cdot U_z{}^\circ) h^2\varphi_{tt} +$$
$$U_z{}^\circ\cdot h^3\nabla\varphi_{tt}\} = O(\varepsilon^3)$$

This equation is a non-local analogue of the Boussinesq equation and reduces to it when $\varepsilon = \sqrt{h}, D = \sqrt{h}D_1$ in the first approximation with respect to h.

It takes account of all dispersion terms as well as the effect of the wind.

2. Weakly non-linear interactions of wind waves in a reservoir with an uneven bottom. Let us find certain asymptotic solutions of Eq.(1.10). When $\varepsilon = 0, \delta = h$ (1.10) admits of solutions in the form of "distorted" plane waves which, as $h \to 0$, are rapidly oscillating functions /10/. The asymptotic forms of such solutions have the form

$$A(x, t) \exp\{iS(x, t)/h\} + o(1)$$

Naturally, a superpositioning of these solutions is also a solution. In particular, the solution of the Cauchy problem with the initial conditions

$$\begin{aligned} &t = 0,\ \varphi = A_1{}^\circ(x) \exp\{iS_0(x)/h\} + \text{c.c.} \\ &h\varphi_t = A_2{}^\circ(x) \exp\{iS_0(x)/h\} + \text{c.c.} \\ &A_j{}^\circ(x) \in C_0{}^\infty(R^2),\ j = 1, 2,\ S_0(x) \in C^\infty(R^2),\ \nabla S_0 \neq 0 \end{aligned} \tag{2.1}$$

is a linear combination of two distorted plane waves travelling in opposite directions.

In Eq.(2.1), c.c. denotes a complex-conjugate expression.

The natural question arises concerning the existence of analogous solutions in the non-linear case and also regarding the law governing the superimpositioning of these solutions. The answer to this question is ambiguous and is determined by the form of the dependence of ε on h. If $\varepsilon = O(h^\alpha)$, then, in the predominant term (apart from corrections $o(1)$ as $h \to 0$), it is found that non-linear effects only play a role when $\alpha \leqslant {}^1/_2$.

In the case when $\alpha = {}^1/_2$ the problem of locating the leading term of the asymptotic form of the Cauchy problem being studied at the times being considered is proper.

The following results hold. The leading term of the asymptotic solution of the Cauchy problem (1.10), (2.1) subject to certain additional assumptions regarding the times $t \in [0, T]$ has the form

$$\begin{aligned} &A_+(x, t) \exp[iS_+(x, t)/h] + \\ &\quad A_-(x, t) \exp[iS_-(x, t)/h] + h^{-1/2} A_0(x, t) \\ &A_\pm(x, t) \in C_0{}^\infty(R^2 \times R^+),\ A_0 S_\pm \in C^\infty(R^2 \times R^+) \\ &S_+ = -S_-,\ A_+ = \bar{A}_-,\ \nabla S_\pm \neq 0 \end{aligned} \tag{2.2}$$

At the same time the equations for the phase $S_\pm$ and the square of the modulus of the amplitude $|A_\pm|^2$ remain the same as in the linear case. The phase is determined from the Hamilton-Jacobi equation with an initial condition corresponding to (2.1):

$$\begin{aligned} &\partial S_\pm/\partial t \pm H(x, \nabla S_\pm) = 0;\ t = 0,\ S_+ = -S_- = S_0(x), \\ &H(x, p) = (|p| \operatorname{th}(|p| D(x)))^{1/2} \end{aligned} \tag{2.3}$$

and is expressed in terms of the solution $X^\pm, P^\pm$ of a system of first-order ordinary differential equations (a Hamiltonian system)

$$x^\cdot = H_p,\ p^\cdot = -H_x;\ t = 0,\ x = \alpha,\ p = \partial S_0(\alpha)/\partial\alpha \tag{2.4}$$

Then, under the assumption that the Jacobian $J = \det|\partial X^\pm/\partial\alpha|$ differs from zero for all $t \in [0, T]$, we have

$$S_\pm(x, t) = \pm S_0(\alpha) + \int_0^t (P^\pm X^{\cdot\pm} - H(X^\pm, P^\pm))\, d\tau$$

The quantity $|A_\pm|^2$ satisfies the equation

$$J^{-1}(d/dt)(J\,|A_\pm|^2) = -2B_I(\omega, p, x, t)\,\omega^3 p^{-2}\,|A_\pm|^2$$

where $\alpha = \alpha^\pm(x, t)$ is the solution of the equation $X^\pm(\alpha, t) = x$, $\alpha \in \operatorname{supp} A_1{}^\circ \cup \operatorname{supp} A_2{}^\circ$, $p = \nabla S_\pm$, $\omega = \partial S_\pm/\partial t$, the function $B_I = \operatorname{Im} B_1$ (see (1.10)) is determined from the Rayleigh

Eq.(3.3) and d/dt is a dertivative along a trajectory $(X\pm, P\pm)$ of system (2.4).

Non-linearity only shows up in a phase correction which is determined from the equation for the amplitudes ((2.8), when $N = 1$). When determining this correction it is additionally necessary to solve the wave equation for A_0 ((2.9) when $N = 1$), the right-hand side of which is proportional to the derivative of the square of the modulus of the amplitude. We note that, in the calculation of the velocity field, the last term in (2.2) gives a correction to the leading term of $O(h^{1/2})$.

Hence, in the case of the Cauchy problem (1.10), (2.1), the leading term of the asymptotic form is determined in a similar manner to that used in the linear case, and non-linearity only introduces a small additive phase correction. At the same time, the phase is substantially dependent on the inhomogeneities of the relief of the bottom while the effect of the wind only has an influence in the determination of the amplitude, which leads, as in the linear approximation, to the appearance of an increment /10/.

Let us now consider the question concerning the superimpositioning of a large number of solutions, that is, for Eq.(1.10), we shall investigate the solution of the Cauchy problem with the initial conditions

$$\begin{aligned} & t = 0,\ \varphi = \varphi_1(x),\ h\varphi_t = \varphi_2(x) \\ & \varphi_j = \Sigma,\ A_k^j(x) \exp\{iS_k^\circ(x)/h\},\ j = 1, 2 \\ & A_k^{1,2} \in C_0^\infty(R^2), \quad S_k^\circ \in C^\infty(R^2), \quad S_k^\circ = -S_{-k}^\circ \\ & A_k^{1,2} = \bar{A}_k^{1,2}, \quad \nabla S_k^\circ \neq 0, \quad k = \pm 1, \ldots, \pm N \end{aligned} \tag{2.5}$$

It is found that, subject to certain additional assumptions (vide infra), the leading term of the asymptotic form of (1.10), (2.5) is representable, as in the linear case, in the form

$$\begin{aligned} & k = -2N, \ldots, 2N,\ k \neq 0 \\ & \Sigma A_k(x, t) \exp\{iS_k(x, t)/h\} + h^{-1/2} A_0(x, t) \end{aligned} \tag{2.6}$$

Each of the phases S_k in (2.6) is, as before, a solution of the Cauchy problem for the Hamilton-Jacobi Eq.(2.3) (cf. (2.5))

$$t = 0,\ S_{2l-1} = -S_{2l} = S_l^\circ(x),\ l = 1, \ldots, N \tag{2.7}$$

which, as previously, is expressed in terms of the solution of a Hamiltonian system which satisfies the conditions

$$\begin{aligned} & t = 0,\ x_{2l-1} = x_{2l} = \alpha,\ p_{2l-1} = p_{2l} = \partial S_l^\circ(\alpha)/\partial\alpha \\ & l = 1, \ldots, N,\ \alpha = (\alpha_1, \alpha_2) \end{aligned}$$

Generally speaking, the equations for the squares of the amplitudes are not obtained by the same means as in the linear case: they are substantially dependent on the occurrence of a resonance interaction between the individual waves. In order to describe this resonance interaction, we introduce into the treatment the functions $S^{mnl} = S_m + S_n + S_l$ with mutually different $n,\ m,\ l \in \{\pm 1, \pm 2, \ldots, \pm 2N\}$ which satisfy one of the initial conditions (2.7) and the sets Γ_{mnl} of the zeros of the functions $\{\partial S^{mnl}/\partial t \pm H(x, \nabla S^{mnl})\}$. The following cases are possible.

1°. For possible sets of m, n, l $\Gamma_{mnl} = \varnothing$. In this case there is no resonance interaction. The equations for the square of the moduli of the amplitudes are defined in a similar manner as for the linear case. Non-linearity solely leads to the occurrence in the leading term of corrections to the phases which are found from equations which are analogous to the case when $N = 1$.

2°. There exist sets of m, n and l such that $\Gamma_{mnl} = R^2$ while the remaining $\Gamma_{mnl} = \varnothing$. We shall refer to such resonances as strong resonances. In this case the resonance interaction leads to the appearance of a new phase $S_k = S^{mnl}$ with a number $k \neq m, n, l$ and non-linearity plays a role in the determination of the leading term of the asymptotic form even in the equations for the squares of the moduli of the amplitudes $|A_k|^2$. The equations for the amplitudes and the smooth component with initial conditions corresponding to (2.5) have the form (a scheme for their derivation is presented in Sect.4)

$$\begin{aligned} & iJ^{-1/2}(d/dt_k)(J^{1/2}A_k) = -\omega_k p_k^{-2}(p_k^2 - B_1(p_k, \omega_k)\omega_k^2 + \\ & \quad U_z^\circ \cdot p_k\omega_k) A_k + {}^1/_2\omega_k^{-1} A_k (\Sigma |A_m|^2 \alpha_m^k(x, t) + 2\omega_k p_k \cdot \nabla A_k + \\ & \quad p_k^2 \operatorname{ch}^{-2}(|p_k|D) A_{0t}) + {}^1/_2\omega_k^{-1}\Sigma'(\beta_{mnl}^k(x, t) A_m A_n A_l), \\ & m = \pm 1, \ldots, \pm 2N \\ & t = 0,\ A_{2l-1} = \bar{A}_{-2l+1} = {}^1/_2(A_l^1 + iA_l^2(H(x, \nabla S_l^0))^{-1}) \end{aligned} \tag{2.8}$$

$$A_{-2l} = \bar{A}_{2l} = {}^1/_2 (A_l^1 - iA_l^2 (H(x, \nabla S_l^\circ))^{-1})$$
$$l = 1, 2, \ldots, N$$
$$A_{0tt} - \nabla\cdot(D\nabla A_0) = \Sigma\{\partial/\partial t(|A_m|^2(\omega_m^4 - p_m^2)) - 2\nabla\cdot(|A_m|^2\omega_m p_m)\} \tag{2.9}$$
$$t = 0,\ A_{0t} = A_0 = 0,\ m = \pm 1, \ldots, \pm 2N$$

It is assumed here, as above, that the Jacobians $J_k = \det|\partial X_k(\alpha, t)/\partial\alpha| \neq 0$ for all $t \in [0, T]$, d/dt_k is a derivative along a trajectory (X_k, P_k) of a Hamiltonian system, $p_k = \nabla S_k$, $\omega_k = \partial S_k/\partial t$, B_1 is defined in (1.10), the summation in Σ' is carried out over such m, n and l for which $\Gamma_{mnl} = R^2$ and the coefficients α_m^k, β_{mnl}^k are determined from the equalities

$$\begin{aligned}
\alpha_m^k &= 2b_{k,-m}\{\omega_k^2\omega_m^2(\omega_k - \omega_m) + G_{k,-m}\omega_m] - (p_k - p_m)\times \\
&\quad (\omega_k p_m + \omega_m p_k) - \omega_k\omega_m^2 G_{k,-m} - p_k\cdot p_m(\omega_k - \omega_m)\} + \\
&\quad {}^1/_2(\omega_k^2\omega_m^2 + p_k^2\omega_m^4 - p_k^2\omega_k^2\omega_m^2 + 3\omega_k^2\omega_m^4 - \omega_k^4 p_m^2 - \\
&\quad p_k^2 p_m^2) - \omega_k^2\omega_m^2 G_{k,-m}(\omega_k^2 - \omega_m^2) \\
b_{km} &= [(\omega_k + \omega_m)(\omega_k^2\omega_m^2 + 2p_k\cdot p_m) + p_k^2\omega_m + p_m^2\omega_k - \\
&\quad G_{m,-k}(\omega_m\omega_k^2 + \omega_k\omega_m^2)][G_{m,k} - (\omega_m + \omega_k)^2]^{-1} \\
G_{m,k} &= |p_m + p_k\,\mathrm{th}(|p_m + p_k|D(x)) \\
\beta_{mnl}^k &= -\omega_k\omega_m^2\omega_l p_n^2 + \omega_k\omega_m^2 G_{l,n}\omega_l\omega_n^2 + \\
&\quad {}^1/_2 p_k\cdot p_m(p_l\cdot p_n + \omega_l^2\omega_n^2) + {}^1/_2\omega_k^2\omega_l^2(p_l\cdot p_n + \omega_l^2\omega_n) - \\
&\quad p_m\cdot\omega_m p_l\omega_l\omega_n^2 + {}^1/_2\omega_m\omega_l\omega_n^2 p_n^2 - \omega_m\omega_n^2\omega_l(p_l + p_n)^2 + \\
&\quad \omega_k^2\omega_m\omega_l\omega_n^2 G_{l,n} - {}^1/_2\omega_k^2\omega_l p_n^2\omega_m + \omega_k p_n\cdot(p_m + p_l)b_{ml} + \\
&\quad \omega_k G_{m,l}b_{ml} + p_k(\omega_m + \omega_l)\cdot p_n b_{ml} + p_k\cdot(p_m + p_l)\omega_n b_{ml} + \\
&\quad \omega_k^2[(\omega_m + \omega_l)\omega_n^2 + G_{m,l}\omega_n]b_{ml}
\end{aligned}$$

3°. There exist m, n and l such that Γ_{mnl} is not identical either with R^2 or with $\varnothing$. This case is only possible if S^{mnl} is non-linearly dependent on x and t which, in particular, is caused by inhomogeneities in the bottom, $D(x)$. We shall refer to such resonances as weak resonances. For an arbitrary Γ_{mnl}, the question regarding the determination of the leading term of the asymptotic solution remains open in this case. However, when Γ_{mnl} is a one-dimensional curve: $\Gamma_{mnl} = \{x = \chi(t, s),\ s \in R^1\}$, it is found that, subject to an additional condition on S^{mnl}, weak resonances do not make any contribution to the leading term of the asymptotic form (see paragraph 4) and the equations for A_k and A_0, as in case (2), have the form of (2.8), (2.9). This condition has the form

$$(\partial/\partial t)\{S_t^{mnl} \pm H(x, \nabla S^{mnl})\} \neq 0 \text{ on } \Gamma_{mnl} \tag{2.10}$$

3. A scheme for eliminating z in order to obtain the equations on the free boundary.

In order to eliminate z and reduce the problem to a system of equations on the free boundary we express Φ_z, w_z, u_z in (1.1), (1.4), when $z = \varepsilon\eta$, in terms of φ, v, ψ, η.

Eqs.(1.2), (1.5) and the condition $\varphi = \Phi|_{z=\varepsilon\eta}$ constitute an elliptic boundary value problem in the layer $-D(x) \leqslant z \leqslant \varepsilon\eta$. By virtue of the assumption concerning the regular dependence on ε, the functions $\Phi = \Phi_0 + \varepsilon\Phi_1 + \ldots$ where Φ_j, $j \geqslant 0$ satisfy a chain of recurrence problems in the layer $-D \leqslant z \leqslant 0$ /13/:

$$\begin{aligned}
&\Phi_{jzz} + h^2\Delta\Phi_j = 0 \\
&z = -D(x),\ \Phi_{jz} + h\nabla D\cdot h\nabla\Phi_j = 0; \\
&z = 0,\ \Phi_j = \varphi_j \\
&(\varphi_0 = \varphi,\ \varphi_j = -\Sigma\eta^{j-m}((j-m)!)^{-1}[\partial^{j-m}\Phi_m/\partial z^{j-m}]|_{z=0}, \\
&m = 0, 1, \ldots, j-1,\ j \geqslant 1)
\end{aligned} \tag{3.1}$$

The solutions of (3.1) are expressed in terms of $\varphi_j\ (j \geqslant 0)$ in the form of an h-pseudo-differential operator

$$\Phi_j = R(x, -ih\nabla, z, h)\varphi_j \tag{3.2}$$

the symbol /14/ of which admits a regular expansion with respect to h as $h \to +0$. The explicit form of the symbol of the operator R is given in /13, 15/.

Similarly, Eqs.(1.3), (1.4) and (1.6) and the conditions $w = v$ and $u = \psi$, when $z = \varepsilon\eta$ constitute a problem in the layer $\varepsilon\eta \leqslant z < \infty$, the solution of which enables one to determine w_z and u_z when $z = \varepsilon\eta$. By virtue of the assumption regarding the regular dependence on ε, we have

$$u = u_0 + \varepsilon u_1 + \ldots,\ w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots$$

The expansion of $u_j, w_j, j \geqslant 0$ in Taylor series in powers of $\varepsilon\eta$ leads to boundary value problems in the even simpler "unperturbed" layer $0 \leqslant z < \infty$. Allowing for the fact that, in order to obtain the solution of the initial problem with an accuracy of $O(\varepsilon^3)$, it is sufficient (see above) to consider the air layer in the quasilinear approximation, we retain solely the terms accompanying ε° from the resulting system (even in the layer $0 \leqslant z < \infty$).

In order to obtain greater accuracy with respect to ε it is necessary to consider terms accompanying ε^k, $k > 0$.

It is sufficient to obtain w_{0z} with an accuracy $O(\varepsilon)$ when $z = \varepsilon\eta$. Here, we assume that the function w_0 in the layer $0 \leqslant z < \infty$ is representable in the form of an h-pseudodifferential operator applied to the function v: $w_0 = R_w(x, t, -ih\nabla, -ih\partial/\partial t, z, h)\, v$. The symbol of this operator, as $h \to 0$, admits of the expansion: $R_w = R_{w0} + (-ih) R_{w1} + \ldots$. The functions R_{wj} $(j \geqslant 0)$ are determined /10/ from a chain of recurrence problems consisting of ordinary differential equations in z and boundary conditions when $z = 0$ and $z \to \infty$. The variables x, p, t and ω occur in these problems as parameters. In particular, R_{w0} is determined from the Rayleigh problem with the parameters x, p, t and ω:

$$(\omega + U\cdot p)\{(R_{w0})_{zz} - |p|^2 R_{w0}\} - U_{zz}\cdot p R_{w0} = 0 \tag{3.3}$$
$$z = 0,\ R_{w0} = 1;\ z \to \infty,\ R_{w0} \to 0$$

4. A scheme for obtaining Eqs.(2.3), (2.8) and (2.9). We obtain Eqs.(2.3)-(2.9) in the following manner. We seek a solution of (1.10) $\varphi + \varphi_1 + \varphi_2 + \ldots$ /5, 6/ and assume that the leading term φ_0 can be represented in the form of a superimpositioning of the individual waves

$$\Sigma A_k \exp\{iS_k/h\},\ k = \pm 1, \ldots \pm N$$
$$S_{-k} = -S_k,\ \bar{A}_k = A_{-k},\ \nabla S_k \neq 0$$

(φ_1 and φ_2 are corrections to φ_0). By substituting φ into (1.10) and equating the coefficients accompanying h° and the powers of h, we obtain Eq.(2.3) for the determining the phases S_k.

For the correction φ_1 we have the equation

$$h^2\partial^2/\partial t^2\varphi_1 + B_0\varphi_1 = ih^{1/2}\,[\Sigma C_{nl}{}^1 A_n A_l \exp\{i(S_n + S_l)/h\}] + (\text{ right-hand side of (2.9)});\ l, n = \pm 1, \ldots \pm N,\ l \neq -n$$

Application of the operator $\{h^2\partial^2/\partial t^2 + B_0\}$ to the first group of terms leads to the multiplication of each term by a quantity of the form $\{(\omega_m + \omega_l)^2 + G_{ml}\}^{-1}$ while application of this operator to the second group of terms leads to the appearance of the smooth component A_0 which satisfies Eq.(2.9). In this case application of the operator to the first group of terms is possible since the equality $S_k = S_m + S_l$ cannot be satisfied (for example, see /2/).

Similarly, let us consider the equation for the correction φ_2. Here, unlike in the previous case, it is now necessary to take account of the resonance terms. From the equation for φ_2 we obtain Eq.(2.8) for the amplitude A_k. The resonance contribution to this equation only introduces those S_k, S_m, S_n, S_l $(k, m, n, l \neq 0,\ k \neq -m,\ k \neq -n,\ k \neq -l,\ m \neq -n,\ m \neq -l,\ n \neq -l)$, for which $S_k = S^{mnl} = S_m + S_n + S_l$ in $R_x{}^2$. In the case of weak resonance it turns out that, subject to condition (2.10), application of the operator $\{h^2\partial^2/\partial t^2 + B_0\}$ leads to the results $o(1)$. In other words, weak resonances do not make a contribution to Eq.(2.8) and this means that they do not make a contribution to the leading term of the asymptotic solution (2.2) /6/.

Actually, let condition (2.10) be satisfied for the function $F = S_n + S_m + S_l$ on the curve $\Gamma \subset R_x{}^3$. Then, the solution of the problem

$$h^2\varphi_{tt} + B_0\varphi = hA(x, t)\exp\{iF(x, t)/h\} \tag{4.1}$$
$$t = 0,\ \varphi = h,\ \varphi_t = 0,\ A \in C_0{}^\infty$$

as $h \to +0$, is the quantity $o(1)$. In fact, following /7/, we represent the solution of problem (4.1) as follows:

$$\varphi = \sum_{+,-} \int_0^t \psi^\pm(x, t, \tau)\exp\{iS^\pm(x, t, \tau)/h\}\,d\tau \tag{4.2}$$

We substitute (4.2) into (4.1) and after differentiating and applying the commutation formula for a h-pseudodifferential operator with an exponent /4/ under the integral sign and equating the coefficients accompanying similar powers of h, we obtain

$$S_t^{\pm} \pm H(x, \nabla S^{\pm}) = 0;\ t = \tau,\ S^{\pm} = F(x, \tau) \tag{4.3}$$
$$(J^{\pm})^{-1/2} (d/dt^{\pm}) [(J^{\pm})^{1/2} \psi^{\pm}] = 0$$
$$t = \tau, \psi^{\pm} = \mp A(x, \tau)(2iH(x, \nabla F))^{-1}$$

Here $X^{\pm}(\alpha, \xi, t), P^{\pm}(\alpha, \xi, t)$ is the solution of Eq.(2.4) with the initial conditions $X^{\pm} = \alpha,\ P^{\pm} = \xi$ when $t = 0$ and $d/dt^{\pm}$ is the derivative along the trajectories $X^{\pm}(\alpha, \nabla F(\alpha, \tau),\ t - \tau),\ P^{\pm}(\alpha, \nabla F(\alpha, \tau),\ t - \tau)$.

In (4.2) let us consider just the term with the plus sign (the term with a minus sign is treated in a similar manner) and apply the stationary-phase method to the integral in (4.2). For this purpose we calculate the derivative $\partial S/\partial\tau$. Let us show that $dS_\tau/dt = 0$. Actually $dS_\tau/dt = S_{\tau t} + H_p \cdot \nabla S_\tau$ but $S_{\tau t} = -H_p(\cdot)\nabla S_\tau$. It is obvious that, when $t = \tau$ $S_\tau = F_t(x, \tau) + H(x, \nabla F(x, \tau))$. Then, in order that $\partial S/\partial\tau$ should vanish at the point (x°, τ°), it is necessary and sufficient that the condition $X(x^\circ, \nabla S(x^\circ, t, \tau^\circ), \tau^\circ - t) \in \Gamma$ should be satisfied at this point.

At the point (x°, τ°) we have

$$\begin{aligned} S_{\tau\tau}(x, t, \tau) = (\partial/\partial\tau)(F_t(X(x, \nabla S,\ \tau - t), \tau) + \\ H(X(x, \nabla S,\ \tau - t),\ P(x, \nabla S,\ t - \tau))) = F_{tt} + \\ \nabla F_t \cdot H_p + H_x(\partial X/\partial\xi)\nabla S_\tau + H_p(\partial P/\partial\xi)\cdot\nabla S_\tau + \\ \nabla F_t(\partial X/\partial\xi)\cdot\nabla S_\tau \end{aligned} \tag{4.4}$$

Here, in the case of the function H and its derivatives, the arguments (X, P) are omitted, the arguments (X, τ) are omitted in the case of the function F, and the arguments $(x, \nabla S, \tau - t)$. in the case of the function X and P.

It follows from (4.4) that, if $|\nabla S_\tau| = 0$ at the point (x°, τ°) then $S_{\tau\tau} \neq 0$ at this point by virtue of condition (2.10).

Whence, according to /7/ (page 659), the estimate

$$\| \varphi \|_{L_2}(R^2) \leqslant \text{const}\ h^{1/2}$$

follows.

Hence, the leading term of the asymptotic form of (2.6) is determined by Eqs.(2.3), (2.8), (2.9) and the initial conditions (2.5).

The authors thank V.P. Myasnikova for discussing this paper.

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Translated by E.L.S.

PMM U.S.S.R.,Vol.51,No.5,pp.634-639,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00
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ON THE SELFOSCILLATORY MODES OF MOTION OF A GAS IN PIPES*

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One-dimensional, non-linear selfexcited oscillations of an ideal gas in pipes are studied. One end of the pipe is closed, and boundary conditions connecting in prescribed manner the incident and reflected Riemann invariants are specified at the other end. Periodic solutions containing shock waves are constructed. A relation connecting the amplitude and the period of the oscillatory motion of the gas is established. The solutions obtained are analysed numerically for stability. The investigations are based mainly on the results of /1-6/ where the forced resonant and subresonant oscillations of a gas in open a closed pipes were studied.

In /1-4/ the equations of oscillations were derived using a method analogous to the Poincaré-Lighthill method of deformed coordinates. The problem was reduced to finding the solutions of ordinary differential equations on the smooth segments, followed by the introduction of discontinuities based on special additional assumptions. In /5, 6/ a sequential approach to solving the class of problems in question was described, within whose framework the problem of discontinuities was solved correctly by analysing the evolution of the compression wave.

The formulation of the boundary value problems in the present paper is related, to a known degree, to the analogous formulations in the investigations of motion of a gas in a Hartman generator where the flows are also oscillatory**. (**A review of such investigations is given in: Dulov V.G. and Maksimov V.P. Thermoacoustics of semiclosed volumes. Preprint 28-86, Novosibirsk, Inst. Theoretical and Applied Mechanics, Siberian Section, Academy of Sciences of the ~~USSR, 1986.). We will use the approach developed in /5, 6/ to analyse the oscillations. The~~ cumbersome derivations given in these papers will be omitted. The arguments concerning the applicability of the isentropic approximation and the possibility of neglecting the change in the Riemann invariants when the characteristics interact with the shock waves, also retain their validity in the case of the oscillations investigated here.

1. Equations of motion.

The equations of gas dynamics in their characteristic form and in the commonly accepted notation are /7/

$$\left(\frac{\partial u}{\partial t}\right)_\xi+\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}\right)_\xi=0,\quad \left(\frac{\partial u}{\partial t}\right)_\eta-\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}\right)_\eta=0,\quad \left(\frac{\partial s}{\partial t}\right)_\zeta=0$$

where the following operators of differentiation along the characteristics C^+, C^-, C° are used:

$$\left(\frac{\partial}{\partial t}\right)_\xi=\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x},\quad \left(\frac{\partial}{\partial t}\right)_\eta=\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x},$$
$$\left(\frac{\partial}{\partial t}\right)_\zeta=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$$

**Prikl.Matem.Mekhan.*,51,5,807-813,1987